Sample Final Exam June 2010

Name: Full Solutions

Part A: Full solutions are not required, but part marks are earned for intermediate steps if the work is shown. Write your answer in the space provided.

1. Evaluate the following limits.

[2]

a) $\lim_{x \to -1} 2 - 3x^2$ $\quad$ -1

b) $\lim_{x \to -\infty} e^x$ $\quad$ 0

2. Determine the derivatives of the following functions. Do **Not** simplify your answers.

a) $y = \left(1 + \sqrt{x^3 + 3}\right)^5$ [3]

$y' = 5(1+\sqrt{x^3+3})^4 \cdot (\tfrac{1}{2}(x^3+3)^{-1/2} \cdot 3x^2$

b) $y = \dfrac{1 - \ln x}{(3x^2 - 5)^7}$ [3]

$y' = \dfrac{(-1/x)(3x^2-5)^7 - (1-\ln x)\cdot 7(3x^2-5)^6 \cdot 6x}{(3x^2-5)^{14}}$

c) $y = (\sin x)^6 (e^{2x})$ [4]

$y' = 6(\sin x)^5 \cdot (\cos x)(e^{2x}) + (\sin x)^6 \cdot e^{2x} \cdot 2$

3. Are the following quantities Scalars (**S**), Vectors (**V**) or Meaningless (**M**)?

a) 100 Nm of Torque is being applied down into the screw. $\quad$ V

b) $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$ $\quad$ S

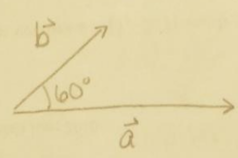
c) $\mathbf{b} - \mathbf{b}$ $\quad$ S

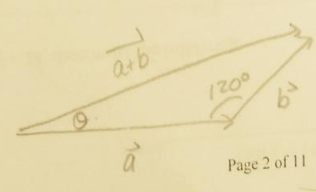
4. Vectors **a** and **b**, with $|\mathbf{a}| = 13$ and $|\mathbf{b}| = 7$, **a** is horizontal and **b** is 60° *above* **a**.

a) $|\mathbf{a} + \mathbf{b}|$ (use cosine law) with 120° $\quad$ ~18

b) The direction of $|\mathbf{a} + \mathbf{b}|$ relative to **a** (use sine law) $\quad$ ~70.3°

c) $\mathbf{a} \bullet \mathbf{b} = |\vec{a}||\vec{b}|\cos 60^\circ$ $\quad$ 45.5





5. List Given vectors $\mathbf{a} = [1, -2, 3]$ and $\mathbf{b} = 3\hat{i} - \hat{j} - 2\hat{k}$ determine the following.

a) $\mathbf{a} \bullet \mathbf{b} = (1,-2,3)\cdot(3,-1,-2)$
$= 3+2-6 = -1$ — -1

b) $\hat{b} = \dfrac{(3,-1,-2)}{|\vec{b}|} = \dfrac{(3,-1,-2)}{\sqrt{14}}$ — $\left(\frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}\right)$

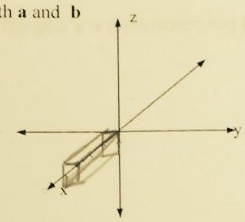
c) the angle between **a** and **b** $\cos\theta = \dfrac{a\cdot b}{|\vec{a}||\vec{b}|}$ — $94°$

d) a vector orthogonal to both **a** and **b** — $(7,11,5)$ $a \times b$

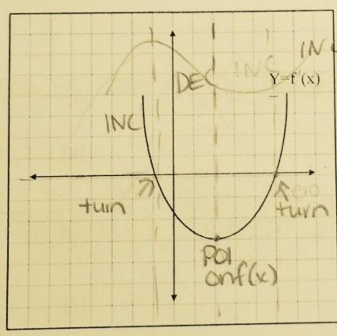
$\begin{matrix} 1 & -2 & 3 \\ 3 & -1 & -2 \end{matrix}$

$a\times b = (4+3,\ 9+2,\ -1+6)$
$= (7,11,5)$

e) Sketch vector **b**



6. Given the graph of f', determine the following intervals:



$f'(x)>0$ when $f(x)$ inc

$f''(x)<0$ conc. down when $f(x)$

$f''(x)=0$ at POI

$f'(x)=0$ at max/min

a) the graph of f is decreasing — $(-1,5)$

[3]

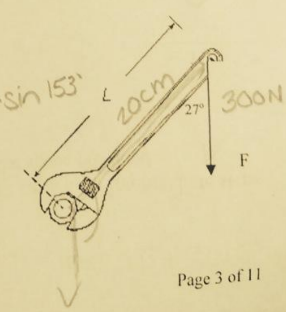
b) the graph of f is concave down — $x<2$

7. a) What is the magnitude of torque produced when a 300 N force is applied at an angle of 27° to a wrench that is 20 cm long? — 2723.94 Nm

$\tau = \vec{r}\times\vec{f}$
$= |\vec{r}||\vec{f}|\sin\theta$
$= 20\cdot 300\cdot\sin 153°$

b) What is the direction of the torque?

into the screw



Name:

8. A function, f, has all the properties listed below. Sketch the graph of $y = f(x)$.

a) $f(x)$ is an odd function, $f(x)$ has no y-intercept

b) $\lim_{x \to +\infty} f(x) = 2$,

[4]

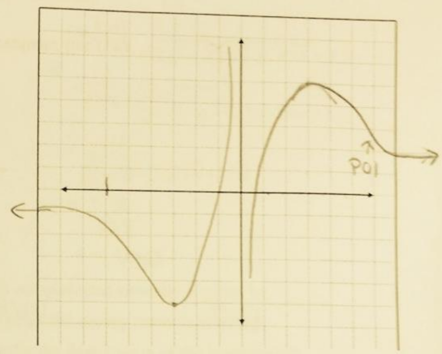
c) $f(3) = 5$, $f'(3) = 0$, $f''(3) < 0$

max/min CD max

d) $f''(x) < 0$ for $0 < x < 6$; and $f''(x) > 0$ for $x > 6$

CD CU

$f''(6)=0$ POI



9. A motorist driving on a straight and level road approaches an intersection. He sees a stop sign and applies his brakes. His car slows down in a way that his velocity, in meters per second, after **t** seconds is $v(t) = 80 - 10t^{\frac{3}{2}}$.

[1] a) Determine the initial velocity of the car.

$v(0) = 80$ m/s

[2] b) How long does it take the car to stop?

$0 = 80 - 10t^{3/2}$

$10t^{3/2} = 80$

$t^{3/2} = 8$

$t = 8^{2/3}$

$t = 4$ sec.

[1] c) Determine the acceleration function of the car.

$a(t) = v'(t) = -15t^{1/2}$

[2] e) Calculate the average acceleration of the car while braking.

avg accel = avg ROC of vel $= \dfrac{v(4) - v(0)}{4}$

$= \dfrac{0 - 80}{4}$

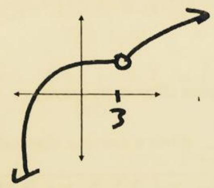
$= -20$ m/s^2

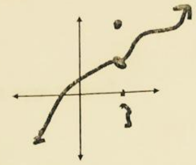
10. Provide a sketch of a function $f(x)$ that is discontinuous at x = 3 because…

a) $f(3)$ does not exist

b) $\lim_{x \to 3} f(x) \neq f(3)$

[2]





11. Determine the following limits **exactly** if they exist, and state if they do not.

$2x^2+x-10$
$= 2x^2+5x-4x-10$
$= x(2x+5)-2(2x+5)$
$= (x-2)(2x+5)$

M-20
A 1
N 5,-4

a) $\lim_{x \to 2} \frac{x^3-8}{2x^2+x-10}$ $= \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(2x+5)} = \frac{12}{9}$ 4/3

b) $\lim_{x \to \infty} \frac{9x^4-3x^7+6}{2x-x^4+6x^7}$ -0.5

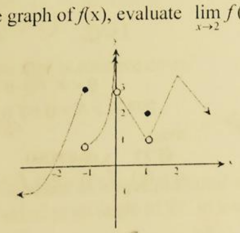
c) $\lim_{x \to 5} \frac{5x-x^2}{|x-5|}$ $= \lim_{x \to 5} \frac{5x-x^2}{-x+5}$ $= \lim_{x \to 5} \frac{x(5-x)}{-x+5} = 5$ 5

d) $\lim_{x \to 0} (1+x)^{\frac{2}{x}}$ 74

e) $\lim_{h \to 0} \frac{(x+h)^{12}-x^{12}}{h}$ $f(x)=x^{12}$ $f'(x)=12x^{11}$ $12x^{11}$

=

f) Given the graph of $f(x)$, evaluate $\lim_{x \to 2} f(x)$ 3.5



g) Given $f(x) = \begin{cases} 2-x & x > 0 \\ \sqrt{x+4} & x = 0 \\ x^2+2 & x < 0 \end{cases}$, evaluate $\lim_{x \to 0} f(x)$ 2

2

2

Sample Final Exam June 2010

Name:_______________

PART B: Full solutions are required. Answer in the spaces provided.

1. Determine the following limit exactly. Show all work. [3]

$$\lim_{x\to 4}\frac{\frac{1}{x}-\frac{1}{2\sqrt{x}}}{x-4}$$

$= \lim_{x\to 4} \dfrac{\frac{2\sqrt{x}-x}{2x\sqrt{x}}}{x-4}$

$= \lim_{x\to 4} \dfrac{2\sqrt{x}-x}{2x\sqrt{x}} \cdot \dfrac{1}{x-4}$

$= \lim_{x\to 4} \dfrac{2\sqrt{x}-x}{(2x\sqrt{x})(x-4)} \cdot \dfrac{2\sqrt{x}+x}{2\sqrt{x}+x}$

$= \lim_{x\to 4} \dfrac{4x-x^2}{(2x\sqrt{x})(x-4)(2\sqrt{x}+x)}$

$= \lim_{x\to 4} \dfrac{-x(-4+x)}{(2x\sqrt{x})(x-4)(2\sqrt{x}+x)}$

$= \dfrac{-4}{2(4)(\sqrt{4})(2\sqrt{4}+4)}$

$= \dfrac{-4}{128}$

$= -1/32$

2. The value of a \$1000 investment earning 6% compounded annually is given by $V = 1000(1.06)^n$, where n is the number of years the money remains invested.

a) What is the average rate of change of value for the first 3 years of the investment?

Avg ROC =

2. The value of a $1000 investment earning 6% compounded annually is given by

$V = 1000(1.06)^n$, where n is the number of years the money remains invested.

a) What is the average rate of change of value for the first 3 years of the investment?

$V'(n) = (0)(1.06)^n + 1000(1.06)^n(\ln 1.06)$
$= 1000(1.06^n)(\ln 1.06)$

$V'(3) = 1000(1.06^3)(\ln 1.06)$
$= 69.4$

$V'(0) = 1000(1.06^0)(\ln 1.06)$
$= 58.3$

Avg ROC $= \frac{V'(0) + V'(3)}{2}$
$= \frac{58.3 + 69.4}{2}$

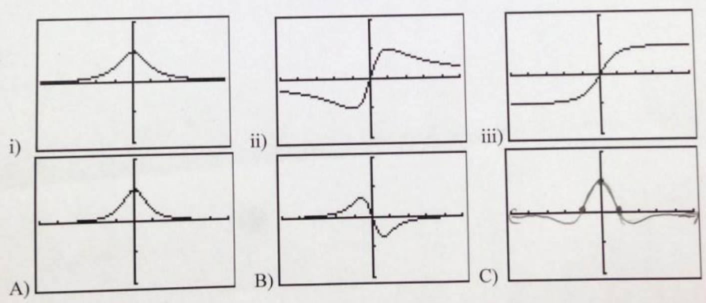
63.8

b) How fast the investment is growing at ten years.

$V'(10) = 1000(1.06^n)(\ln 1.06)$
$= 1000(1.06^{10})(\ln 1.06)$

[4] $= 104.4$

3. Match each given function to its derivative below. Give one reason for each choice. Sketch the unmatched derivative.



[5] i-B | iii-C | ii-A

Increasing till 0, Decreasing after 0 | always increasing | decreasing to point, then inc, then Dec

\

4. Determine and simplify the derivative of $f(x) = \frac{3}{1-2x}$ from first principles.

[5]

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$= \left(\frac{3}{1-2(x+h)}\right) - \left(\frac{3}{1-2x}\right)$$

$$= \left(\frac{3(1-2x)}{1-2x-2h(1-2x)}\right) - \left(\frac{3(1-2x-2h)}{1-2x(1-2x-2h)}\right)$$

$$= \frac{3-6x-(3-6x-6h)}{(1-2x-2h)(1-2x)}$$

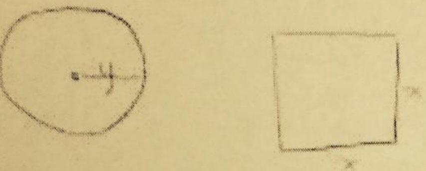
$$= \frac{6h}{(1-2x-2h)(1-2x)} \div \frac{h}{1}$$

$$= \frac{6\cancel{h}}{(1-2x-2h)(1-2x)} \times \frac{1}{\cancel{h}}$$

$$= \frac{6}{(1-2x)(1-2x-2h}$$

5. A piece of string 100cm long is to be cut into two pieces. One piece will be bent into a circle and the other will be bent into a square. Where should the string be cut in order to minimize the total area of the two figures. Verify that your answer is indeed a minimum. (Answer to two decimal places)

[6]



$C = 2\pi y$ $\qquad A = x^2$

$4x + 2\pi y = 100$

$x = \frac{100}{4} - \frac{2\pi y}{4}$

$= 25 - 0.5\pi y$

$A = x^2 + \pi y^2$

$= (25 - 0.5\pi y) + \pi y^2$

$0 = 625 - 25\pi y + \frac{1}{8}\pi^2 y + 2\pi y$

$y\pi(\frac{1}{8}\pi + 2) = 25\pi$

$y = \frac{25}{\frac{1}{8}\pi + 2}$

$\doteq 10$

$C = 2\pi r$

$= 2\pi(10)$

$\boxed{y \doteq 63}$

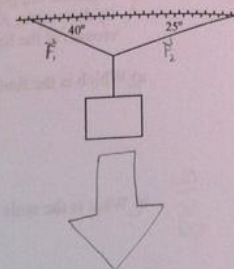
$100 - 63 \doteq x$

$\boxed{37 \doteq x}$

Page 6 will be posted tomorrow!

8. A 100 kg weight is held from the ceiling by two cables, as shown. If the system is perfectly balanced, determine the tension (One decimal place) in each cable.

[5]



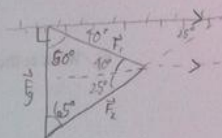
$|\vec{F_g}| = 100kg \times 9.8 N/kg$
$= 980 N$

$\frac{|\vec{F_1}|}{\sin 65^\circ} = \frac{|\vec{F_g}|}{\sin(40+25)^\circ}$

$|\vec{F_1}| = \frac{|\vec{F_g}| \sin 65^\circ}{\sin 65^\circ}$
$= |\vec{F_g}|$
$= 980.0N$

$\frac{|\vec{F_2}|}{\sin 50^\circ} = \frac{|\vec{F_g}|}{\sin(40+25)}$

$|\vec{F_2}| = \frac{\vec{F_g} \sin 50^\circ}{\sin 65^\circ}$
$= \frac{980N \cdot \sin 50^\circ}{\sin 65^\circ}$
$\doteq 828.3 N$



∴ The tension in each cable is 980.0N and 828.3N.

9. Find the scalar equation of the plane that contains the point (2, -1, 5) as well as the line $\langle x, y, z\rangle = \langle 1, 2, 3\rangle + t\langle 3, 4, -3\rangle$.

[5] $P_1(2, -1, 5)$
$P_2(1, 2, 3)$
$\vec{d_1} = \langle 3, 4, -3\rangle$

$\vec{d_2} = P_2 - P_1$
$= \langle 1-2, 2-(-1), 3-5\rangle$
$= \langle -1, 3, -2\rangle$

$\vec{n} = \vec{d_1} \times \vec{d_2}$
$= \langle 3, 4, -3\rangle \times \langle -1, 3, -2\rangle$
$= \langle (4)(-2)-(3)(-3), (-1)(-3)-(3)(-2), (3)(3)-(-1)(4)\rangle$
$= \langle 1, 9, 13\rangle$
A B C

$Ax + By + Cz + D = 0$
$P_1(2, -1, 5) \rightarrow 1x + 9y + 13z + D = 0$
$1(2) + 9(-1) + 13(5) + D = 0$
$2 - 9 + 65 + D = 0$
$58 + D = 0$
$D = -58$

$\boxed{x + 9y + 13z - 58 = 0}$

10. Solve and classify each intersection. [10]

a) $[x,y,z] = [0,-8,4] + t[3,1,-1]$ and $\frac{x-3}{1} = \frac{y+7}{-2} = \frac{z-5}{4}$

① $\vec{r}_1 = [0,-8,4] + t(3,1,-1)$ $\quad \vec{n}_1 = (3,1,-1)$
$\vec{r}_2 = (3,-7,5) + s(1,-2,4)$ $\quad \vec{n}_2 = (1,-2,4)$ } not ∥!

$x_1 = 3t$	$x_2 = 3+s$
$y_1 = -8+t$	$y_2 = -7-2s$
$z_1 = 4-t$	$z_2 = 5+4s$

$x_1 = 1$	$x_2 = 3\frac{1}{3}$
$y_1 = -7\frac{2}{3}$	$y_2 = -7\frac{2}{3}$
$z_1 = 3\frac{2}{3}$	$z_2 = 6\frac{1}{3}$

② $3t = 3+s$
$s = t$
$-8+t = -7-2(t)$
$t = 1-2t$
$3t = 1$
$t = \frac{1}{3}$
$s = \frac{1}{3}$

∴ the lines are skew b/c they dont intersect.

b) $x + 2y + 3z + 4 = 0$
$x - y - 3z - 8 = 0$
$x + 5y + 9z + 16 = 0$

①: $x+2y+3z+4=0 \longrightarrow n_1: (1,2,3)$
②: $x-y-3z-8=0 \longrightarrow n_2: (1,-1,-3)$ ∴ no planes are parallel
③: $x+5y+9z+16=0 \longrightarrow n_3: (1,5,9)$

$\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$
$(1,2,3) \cdot [(1,-1,-3) \times (1,5,9)] = 0$
$(1,2,3) \cdot (6,6,-6) = 0$
$(6+12-18) = 0$
$0 = 0$

$\begin{matrix} 1 & -1 & -3 \\ 1 & 5 & 9 \end{matrix}$
$[(-1)(9)-(5)(-3),\ (-3)(1)-(9)(1),\ (1)(5)-(1)(-1)]$
$(6,6,-6)$

∴ these 3 planes ARE Co-planar (must be "Star" or "Triangle case")

① $x+2y+3z = -4$
$-$ ② $x-y-3z = 8$
④ $3y+6z = -12$

① $x+2y+3z = -4$
$-$ ③ $x+5y+9z = -16$
⑤ $-3y-6z = 12$

④ $3y+6z = -12$
$+$ ⑤ $-3y-6z = 12$
$\boxed{0y = 0}$ ← ∴ "Star" Case

*Find the equation of the line it intersects

$3y+6z = -12$
$3y = -12-6z$
$y = \frac{-12}{3} - \frac{6z}{3}$
$\boxed{y = -4-2z}$

let $z = t$
$x = -2y-3z-4$
$x = -2(-4-2t)-3(t)-4$
$x = 8+4t-3t-4$
$x = 4+t$

$\boxed{\begin{aligned} x &= 4+t \\ y &= -4-2t \\ z &= t \end{aligned}}$

10. Solve and classify each intersection. [10]

a) $[x,y,z] = [0,-8,4] + t[3,1,-1]$ and $\frac{x-3}{1} = \frac{y+7}{-2} = \frac{z-5}{4}$

$x = 3t$ $\qquad$ $x = 3+s$

$y = -8+t$ $\qquad$ $y = -7-2s$

$z = 4-t$ $\qquad$ $z = 5+4s$

$3t = 3+s$

$s = 3t-3$

$-8+t = -7-2(3t-3)$

$-8+t = -7-6t+6$

$-7 = -7t$

$t = 1$

$3t = 3+s$

$3(1) = 3+s$

$s = 0$

$(3,-7,3)$ $\qquad$ $(3,-7,5)$ $\qquad$ $\therefore$ skewed

b) $x + 2y + 3z + 4 = 0$
$x - y - 3z - 8 = 0$
$x + 5y + 9z + 16 = 0$

$$\begin{array}{l} R_1 - R_2 \\ R_3 - R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ 1 & -1 & -3 & 8 \\ 1 & 5 & 9 & -16 \end{array}\right]$$

$$R_3 - 2R_1 \left[\begin{array}{ccc|c} 0 & 3 & 6 & -12 \\ 1 & -1 & -3 & 8 \\ 0 & 6 & 12 & -24 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 3 & 6 & -12 \\ 1 & -1 & -3 & 8 \\ 0 & 0 & 0 & -24 \end{array}\right] \text{parallel}$$

$\therefore$ infinite solutions.

$\therefore (a,b,c) \begin{cases} x = t \\ y = 4-2t \\ z = t-4 \end{cases}$

$x - y - 3z = 8$

$y + 2z = -4$

$x - (-4-2z) - 3z = 8$

$x - z = 4$

$t - z = 4$

$t - 4 = z$

$x - z = 4$

$x - (t-4) = 4$

$x = t$

$-4 - 2(t-4)$

$-4 - 2t + 8$

$4 - 2t = y$

11. Analyze the function $y = \frac{x^2+1}{(x-1)^2}$, under the headings: Domain, Intercepts, Asymptotes, Increasing/Decreasing, Maximum/Minimum, Concavity and Points of Inflection. Then sketch the curve.

[13]

Domain

$x \in R$

Intercepts

$f(0) = 1$

y-int = 1

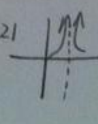
$0 = \frac{x^2+1}{(x-1)^2}$

= Impossible

x-int = none

ASSymtotes

HA: $y=1$ $f(0.9)=181$ $f(1.1)=221$



VA: $x=1$

Inc/Dec

$f(x) = \frac{x^2+1}{(x-1)^2}$

$f'(x) = \frac{-2x-2}{(x-1)^3}$

$0 = \frac{-2x-2}{(x-1)^3}$

$x = -1$

$x \neq 1$

	$-2x-2$	$(x-1)^3$	$f'(x)$	$f(x)$
$(-\infty,-1)$	+	−	−	↓
$(-1,1)$	−	−	+	↑
$(1,\infty)$	−	+	−	↓

} minimum

} * assymtote

Increasing on $(-1,1)$

Decreasing on $(-\infty,-1)(1,\infty)$

Max/Min

Min: $x=-1$ $(-1,0.5)$

$f(-1) = 0.5$

Concavity

$f'(x) = \frac{-2x-2}{(x-1)^3}$

$f''(x) = \frac{4x+8}{(x-1)^4}$

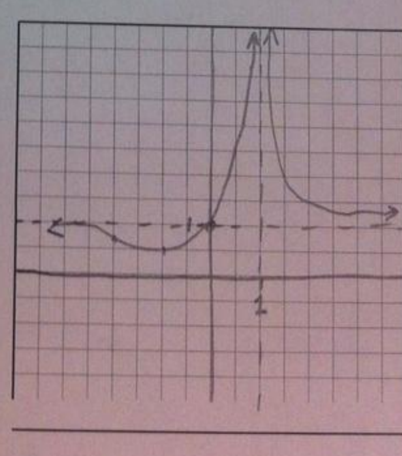
$0 = \frac{4x+8}{(x-1)^4}$

$x = -2$

$x \neq 1$

	$4x+8$	$(x-1)^4$	$f''(x)$	$f(x)$
$(-\infty,-2)$	−	+	−	CD
$(-2,1)$	+	+	+	CU
$(1,\infty)$	+	+	+	CU

} POI $(-2, \frac{5}{9})$



Name: your fave student (Erin)

11. Analyze the function $y = \dfrac{x^2+1}{(x-1)^2}$, under the headings: Domain, Intercepts, Asymptotes, Increasing/Decreasing, Maximum/Minimum, Concavity and Points of Inflection. Then sketch the curve.

[13]

DOMAIN

$\{x \in \mathbb{R} \mid x \neq 1\}$

INTERCEPTS

x-int: NONE

y-int: (0, 1)

SYMMETRY

neither

MAX/MINS

$$y' = \frac{2x(x-1)^{\cancel{2}} - (x^2+1)(2)\cancel{(x-1)}(1)}{(x-1)^{\cancel{4}3}}$$

$$= \frac{2x(x-1) - 2(x^2+1)}{(x-1)^3}$$

$0 = \cancel{2x^2} - 2x - \cancel{2x^2} - 2$

$0 = -2x - 2$

$0 = -2(x+1)$

$x = -1$

Int	-2	x+1	$(x-1)^3$	f'	∴ f(x)
$(-\infty, -1)$	−	−	−	−	dec > min
$(-1, 1)$	−	+	−	+	inc
$(1, \infty)$	−	+	+	−	dec

min @ $(-1, \frac{1}{2})$

ASYMPTOTES

VA @ x = 1

HA @ y = 1

CONCAVITY

$$y'' = \frac{-2(x-1)^{\cancel{3}} - (3)\cancel{(x-1)^2}(-2x-2)}{(x-1)^{\cancel{6}4}}$$

$$= \frac{-2x + 2 + 6x + 6}{(x-1)^4}$$

$0 = 4x + 8$

$-8 = 4x$

$x = -2$

Int	4x+8	$(x-1)^4$	f''	∴ f(x)
$(-\infty, -2)$	−	+	−	CD > POI
$(-2, 1)$	+	+	+	CU
$(1, \infty)$	+	+	+	CU

POI @ $(-2, 5/9)$