Lesson 3: The Second Derivative: Concavity

For Concavity: Concave Up when $f''(x) > 0$
Concave Down when $f''(x) < 0$

A **<u>Point of Inflection</u>** occurs when concavity changes.

Example 1: Discuss the concavity of

a) $f(x)=\dfrac{x}{x^2+1}$

$f(-\sqrt{3})=-\dfrac{\sqrt{3}}{4}$

$f(0)=0$

$f(\sqrt{3})=\dfrac{\sqrt{3}}{4}$

$$f'(x)=\frac{(x^2+1)-x(2x)}{(x^2+1)^2}$$

$$=\frac{-x^2+1}{(x^2+1)^2}$$

$$f''(x)=\frac{(-2x)(x^2+1)^2-(-x^2+1)(2)(x^2+1)(2x)}{(x^2+1)^4}$$

$$0=2x(x^2+1)\left[-x^2\cancel{-}1-(-x^2+1)(2)\right]$$

$$=2x(x^2+1)(\cancel{2}x^2-3)$$

$x=0$

$\cancel{3}x^2-\cancel{3}=0$

$x^2=3$

$x=\pm\sqrt{3}$

Interval	$2x$	x^2+1	x^2-3	f''	Conc.
$(-\infty,-\sqrt{3})$	−	+	+	−	C.D.
$(-\sqrt{3},0)$	−	−	+	+	C.U.
$(0,\sqrt{3})$	+	−	+	−	C.D.
$(\sqrt{3},\infty)$	+	+	+	+	C.U.

POI

$\left(-\sqrt{3},-\dfrac{\sqrt{3}}{4}\right)$

$(0,0)$

$\left(\sqrt{3},\dfrac{\sqrt{3}}{4}\right)$

Example 1: Discuss the concavity of

b) $f(x) = e^{-\frac{1}{2}x^2}$

$f(-1) = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

$f(1) = \frac{1}{\sqrt{e}} = \frac{\sqrt{e}}{e}$

$f'(x) = e^{-\frac{1}{2}x^2} \cdot (-x)$

$f''(x) = e^{-\frac{1}{2}x^2} \cdot x^2 - e^{-\frac{1}{2}x^2}$

$= e^{-\frac{1}{2}x^2}(x^2 - 1)$

$= e^{-\frac{1}{2}x^2}(x-1)(x+1)$

$x = 1 \quad x = -1$

Interval	$e^{-\frac{1}{2}x^2}$	$x-1$	$x+1$	f''	Conc
$(-\infty, -1)$	+	−	−	+	C.U.
$(-1, 1)$	+	−	+	−	C.D.
$(1, \infty)$	+	+	+	+	C.U.

P.O.I @ $\left(-1, \frac{1}{\sqrt{e}}\right)$
$\left(1, \frac{1}{\sqrt{e}}\right)$