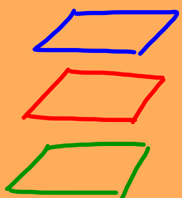
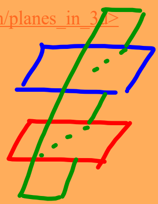
Lesson 7: Intersecting Three Planes

With 3 planes what are the possibilities for an intersection?
(Hint: 4 cases for a solution, 4 cases for no solution)

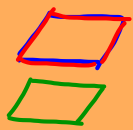
<http://www.josechu.com/planes_in_3d>



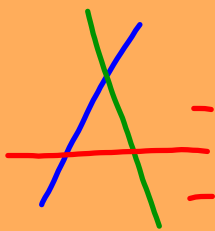
→ 3 parallel planes
→ By inspection
→ No solution



→ 2 planes are parallel
→ 3 one cuts through
→ no solution
→ By inspection

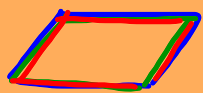


→ 2 coplanar
→ 3rd parallel
→ Solve by inspection
→ No Solution

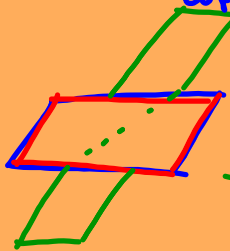


- Triangle case
- no solution
- Solve w̄ algebra

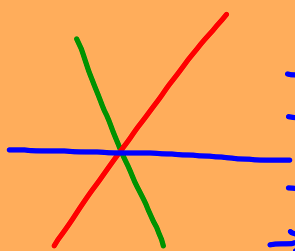
• coplanar



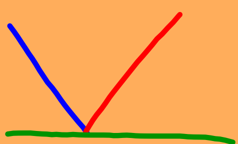
→ 3 coplanar planes
→ the plane is the solution
→ by inspection



→ 2 coplanar/coincident
→ 3rd cuts through
→ intersection is a line



→ Star case
→ planes intersect in a line
→ none are coplanar
→ Solve w̄ algebra



- intersect at a point
- solve by algebra

Strategies

→ look for parallel planes (distinct or coplanar)

→ Coplanar <u>iff</u> $a \cdot (b \times c) = 0$

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Ex 3: **Describe** the intersection of the given planes.
Solve the intersection.

$\pi_1 : x - 3y + 3z = -4$

$\pi_2 : z = 2x + 3y - 15$ $\quad 2x + 3y - z = 15$

a) $\pi_3 : z = 4x - 3y - 19$ $\quad 4x - 3y - z = 19$

$\therefore$ Not parallel

Coplanar:

$$(1,-3,3)\cdot\left((2,3,-1)\times(4,-3,-1)\right) \stackrel{?}{=} 0$$

$$(1,-3,3)\cdot(-6,-2,-18)$$

$$= -6 + 6 - 54$$

$$\neq 0$$

$\therefore$ Not coplanar

$\langle -6,-2,-18\rangle$

use ① + ②

$$\begin{array}{r} x - 3y + 3z = -4 \\ +\ 2x + 3y - z = 15 \\ \hline 3x + 2z = 11 \end{array}$$

② + ③

$$\begin{array}{r} 2x + 3y - z = 15 \\ +\ 4x - 3y - z = 19 \\ \hline 6x - 2z = 34 \end{array}$$

$$\begin{array}{r} 3x + 2z = 11 \\ +\ 6x - 2z = 34 \\ \hline 9x = 45 \end{array}$$

$x = 5$

$3(5) + 2z = 11$

$z = -2$

$(5) - 3y + 3(-2) = -4$

$-3y = -3$

$y = 1$

$\therefore$ These 3 planes intersect at a point $(5, 1, -2)$.

Solve using matrices:

$$\left[\begin{array}{ccc|c} 1 & -3 & 3 & -4 \\ 2 & 3 & -1 & 15 \\ 4 & -3 & -1 & 19 \end{array}\right] \overset{R_1+R_2}{\sim} \left[\begin{array}{ccc|c} 3 & 0 & 2 & 11 \\ 2 & 3 & -1 & 15 \\ 4 & -3 & -1 & 19 \end{array}\right]$$

$$\overset{R_2-R_3}{\sim} \left[\begin{array}{ccc|c} 3 & 0 & 2 & 11 \\ -2 & 6 & 0 & -4 \\ 4 & -3 & -1 & 19 \end{array}\right] \overset{2R_3+R_2}{\sim} \left[\begin{array}{ccc|c} 3 & 0 & 2 & 11 \\ -2 & 6 & 0 & -4 \\ 6 & 0 & -2 & 34 \end{array}\right]$$

$$\overset{R_1+R_3}{\sim} \left[\begin{array}{ccc|c} \cancel{9}\,1 & 0 & 0 & \cancel{45}\,5 \\ -2 & 6 & 0 & -4 \\ 6 & 0 & -2 & 34 \end{array}\right] \overset{R_3-6R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ -2 & 6 & 0 & -4 \\ 0 & 0 & \cancel{-2}\,1 & \cancel{4}\,-2 \end{array}\right]$$

$$\overset{R_2+2R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & \cancel{6}\,1 & 0 & \cancel{6}\,1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$\therefore$ The point is $(5, 1, -2)$

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b) $\pi_1: x + y - z - 5 = 0$

$\pi_2: 2x + y + z = 4$

$\pi_3: x + 2z = 8$

- not parallel
- coplanarity?

$\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) \stackrel{?}{=} 0$

$= 0$

$\therefore$ coplanar

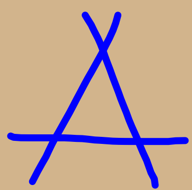
① + ②

$\boxed{3x + 2y = 9}$

$2R_2 - R_3$

$- \boxed{\cancel{3x + 2y = 0}}$

$0y = 9$



$\therefore \Rightarrow$ no solution

$\pi_1 : 7x + y - 3z = 5$

$\pi_2 : x - y - z = -1$

c) $\pi_3 : x + y = 2$

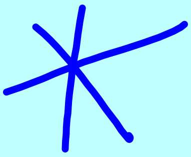
parallel? NO

coplanar? NO

$$\left[\begin{array}{ccc|c} 7 & 1 & -3 & 5 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2 \end{array}\right] \underset{R_1+R_2}{\sim} \left[\begin{array}{ccc|c} 8 & 0 & -4 & 4 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2 \end{array}\right]$$

$$\underset{R_2+R_3}{\sim} \left[\begin{array}{ccc|c} 8 & 0 & -4 & 4 \\ 2 & 0 & -1 & 1 \\ 1 & 1 & 0 & 2 \end{array}\right] \underset{R_1-4R_2}{\sim} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 1 \\ 1 & 1 & 0 & 2 \end{array}\right]$$

$\therefore$ They intersect in a line



$\boxed{x = t}$

$2t - z = 1$

$\boxed{z = 2t - 1}$

$t + y = 2$

$\boxed{y = 2 - t}$

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Ex 4: Give 2 planes (scalar form) that intersect in l_1 : $[x, y, z] = [1, 2, -1] + t[-1, 1, 2]$.